

# Regulatory Capital for Market and Credit Risk Interaction: Is Current Regulation Always Conservative?\*

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## Abstract

In the work of the Basel Committee there has been a tradition of distinguishing market from credit risk and to treat both categories independently in the calculation of risk capital. In practice positions in a portfolio depend simultaneously on both market and credit risk factors. In this case, an approximation of the portfolio value function splitting value changes into a pure market risk plus pure credit risk component can lead to underestimation of risk. It can therefore not be argued that the current regulatory approach would always be conservative from a risk assessment perspective. We discuss this fact in the context of foreign currency loans and argue that under the traditional regulatory approach the true risk of a portfolio of foreign currency loans would be significantly underestimated.

**Keywords:** integrated analysis of market and credit risk, risk management, foreign currency loans, banking regulation.

**JEL-Classification Numbers:** G28, G32, G20, C15.

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# 1 Introduction

The distinction between market and credit risk and their independent analysis has a certain tradition in banking regulation, in particular in the past work of the Basel Committee. Regulators have traditionally thought of credit risk as mainly relevant for the banking book and market risk as mainly relevant for the trading book. In this way the regulatory categorization mimics the traditional organization of banks into a credit department and a market investment department.

When we leave aside operational risk, Pillar 1 of Basel II requires separate regulatory capital for credit and market risk:

$$RC_c + RC_m. \tag{1}$$

Regulatory capital for credit risk,  $RC_c$ , at the moment is calculated for each loan separately, either according to the standard approach or to the IRB approach. Portfolio credit risk models at the moment are not admitted for the calculation of regulatory capital, but they also fit this scheme as long as they assume market risk factors to be deterministic. Regulatory capital for market risk,  $RC_m$  is intended to provide against adverse moves in market prices and do not take into account the possibility of counterparty default.

The separate calculation of regulatory capital in eq. (1) follows the separation into banking book and trading book only roughly. Typically, credit risk is associated to the banking book and market risk is associated to the trading book. But for some positions in the trading book (OTC derivatives, repo-style transactions etc.) regulatory capital for counterparty risk is required by the Basel Committee on Banking Supervision [2005, par. 702–718]. On the other hand, FX risk is calculated not only for the trading book but also for the banking book. And interest risk in the banking book may require additional capital under Pillar 2.

Still, the rough association of credit risk to the banking book and market risk to the trading book may have inspired arguments to the effect that current regulation as expressed in eq. (1) is conservative. Implicitly these arguments have the following pattern:

**Premise 1** ‘Diversification’: Under a subadditive risk measure the risk of the total portfolio will be smaller or at most equal to the sum of the risk of the banking book and of the trading book.

**Premise 2** Credit risk is just relevant to the banking book and market risk is just relevant to the trading book.

**Conclusion** Under all subadditive risk measures total risk will be smaller or at most equal to the sum of market risk and credit risk.

This is a valid argument. If the premises are true the conclusion must necessarily be true. The conclusion can be wrong only if at least one of the

premises is wrong. Premise 1 is not disputable; it is the definition of sub-additivity. Premise 2 is usually not accepted literally, but it is considered a good approximation. So the Conclusion need not necessarily be true—at least not by virtue of the argument. Still it is very popular. Regulation is widely considered conservative because it requires separate risk capital for market and for credit risk. Indeed, if a deviation from eq. (1) is considered, it is only in the direction of reducing capital requirements.

We will show in Section 2 that the inverse of the above argument also holds. Assuming Premise 1, the Conclusion holds *only* if Premise 2 holds. Only if the portfolio is separable into a market subportfolio depending just on market but not on credit risk factors, and credit subportfolio depending just on credit but not on market risk factors, will integrated risk capital be smaller than the sum of market and credit risk capital. In other words, underestimation of risk is possible if the portfolio is not separable into a market and a credit subportfolio.

In this paper we challenge the traditional view that integrated risk capital will always be smaller than the sum of market and credit risk capital. We reject this conclusion both in its literal form and as an approximation. We argue that in many situations a split into credit and market portfolio is not possible because positions in the portfolio will *simultaneously* depend on market and credit risk factors. If in such a situation a subportfolio construction along the traditional lines is enforced this will necessarily lead to wrong portfolio valuation and as a consequence to wrong assessment of the true portfolio risk. Using the example of foreign currency loans we show that under the current regulatory concepts we could have a strong underestimation of the true risk of such a portfolio.

**Related research** The literature on integration of market and credit risk seems to take different perspectives on the risk integration problem. There is one strand of literature that takes a critical view of the traditional categorization. Jarrow and Turnbull [2000] is an early paper that develops a reduced form model for incorporating stochastic interest rates into traditional credit risk models. Medova and Smith [2005] develop a credit risk framework that incorporates stochastic interest rates but is based on a structural credit risk model. Barnhill and Maxwell [2002] propose a simulation framework for an integrated market and credit risk analysis for fixed income portfolios. In contrast to these papers, which all concentrate on modelling issues, our paper works with a model that is stripped down to the conceptual essentials but focuses on the aspect of comparing risk assessment under an integrated analysis with the traditional analysis in which risks are separately analyzed along the lines of the regulatory tradition.

Duffie and Singleton [2003, chap. 13] report on Duffie and Pan [2001] and compare pure market risk (in the absence of credit risk) to integrated

risk of a loan portfolio and find that integrated risk is higher than pure credit risk. In contrast this paper compares integrated risk to the sum of pure market risk and pure credit risk.

Another strand of the recent literature (see Rosenberg and Schuermann [2006], Dimakos and Aas [2004]) about integrated risk modelling seems to take a different perspective. These papers do not take issue with the traditional categorization but rather point out that the portfolios analyzed under the different categories market and credit risk, can be understood as risks of subportfolios of the total bank portfolio. Clearly when subportfolios can be constructed the only issue that remains to be discussed is quantifying the diversification effect if these subportfolios are merged into an overall portfolio. This is exactly what these authors do in their papers. In contrast we argue that the issue of an integrated market and credit risk analysis is not a diversification issue. The problem is often that the subportfolio construction along market and credit risk factors is not possible. If this is the case this fact has to be analyzed head on. If instead in such a situation the portfolio value is approximated by subportfolios of market and credit risk, a valuation error will usually lead to a risk assessment error and if worse comes to worst to a significant underestimation of the true risk.

The paper is organized as follows. Section 2 gives a theoretical analysis where the traditional approach is contrasted with an integrated analysis, Section 3 analyzes foreign currency loans by means of a toy example, Section 4 extends the toy model to a real world simulation of a hypothetical Swiss Franc foreign currency loan portfolio. Section 5 concludes. All proofs are collected in the Appendix.

## **2 Integrated versus separate analysis of market and credit risk**

Current regulation is conceptually based upon the distinction between market and credit risk. Market risk is defined as the risk that a financial position changes its value due to the change of an underlying market risk factor, like a stock price, an exchange rate or an interest rate. Credit risk is defined analogously as the risk of not receiving the promised payment on an outstanding claim. Market risk factors, the determinants of market risk, are usually market prices, or are derived from them. Credit risk factors, the determinants of the components of default losses, like default probabilities, losses given default, exposures at default, may be idiosyncratic properties of individual obligors, or macroeconomic and market variables influencing all obligors in the same way. Some risk factors may influence both market and credit risk. Interest rates, for example, are market prices determining the values of various fixed income instruments, but they also have an influence on default probabilities, and they are in turn influenced by idiosyncratic

properties of individual obligors.

Assume a separation of risk factors into market and credit risk factors is given. It is not important for our argument which risk factors are actually seen as market or as credit risk factors. What matters is that one such separation is made.

Risk assessment is based on portfolio valuation. Let us thus start with this aspect first. Assume a function  $V : \Omega_A \times \Omega_E \rightarrow \mathbb{R}$  is given, which specifies the value of a portfolio in dependence of some vectors  $a \in \Omega_A$  and  $e \in \Omega_E$  of credit and market risk factors, respectively.

*Market risk* deals with the value change of a portfolio which arises from moves in market risk factors, assuming that credit risk factors are constant at some  $a_0$ :

$$\Delta M(e) := V(a_0, e) - V(a_0, e_0).$$

Value changes are calculated in comparison to the portfolio value  $V(a_0, e_0)$  in some reference scenario  $(a_0, e_0)$ . *Credit risk* deals with value changes caused by moves in credit risk factors, assuming all market risk factors are constant at  $e_0$ :

$$\Delta C(a) := V(a, e_0) - V(a_0, e_0).$$

*Integrated risk* is related to the value change caused by simultaneous moves of market and credit risk factors:

$$\Delta V(a, e) := V(a, e) - V(a_0, e_0).$$

Adding up regulatory capital for market and credit risk implicitly rests on the assumption that integrated value changes of the portfolio are approximated by the sum of market plus credit risk factor related value changes:

$$\Delta V(a, e) \approx \Delta C(a) + \Delta M(e). \quad (2)$$

This corresponds to the approximation

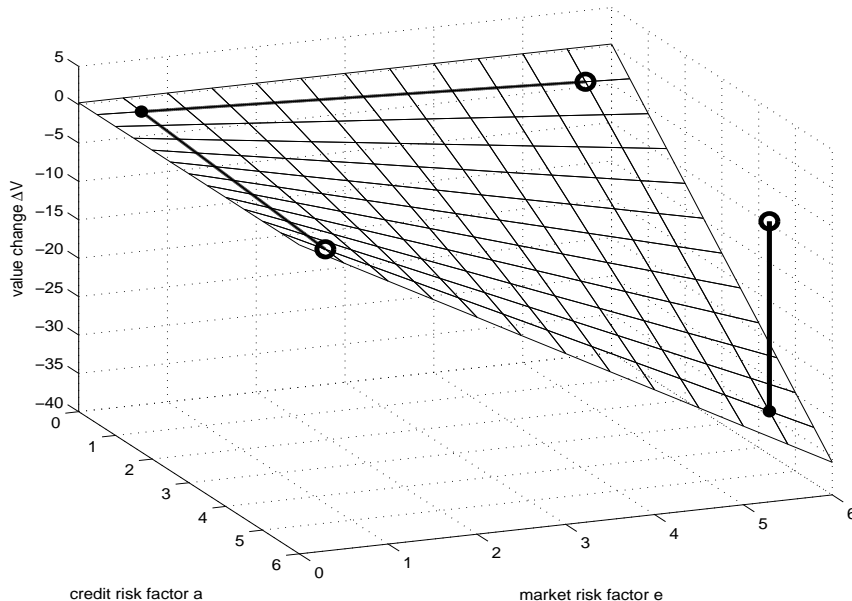
$$V(a, e) \approx V(a_0, e_0) + \Delta C(a) + \Delta M(e)$$

Clearly for a general portfolio valuation function  $V(a, e)$  the approximation  $\Delta C(a) + \Delta M(e)$  not always overestimates but sometimes underestimates the true integrated  $\Delta V$ . If in some scenario  $(a, e)$  the approximation error

$$D(a, e) := \Delta V(a, e) - \Delta C(a) - \Delta M(e)$$

is negative, we have *malign* risk interaction. If  $D$  is non-negative in all scenarios, we say we have *benign* interaction of credit and market risk.

Figure 1 shows a situation with  $D < 0$  where true integrated risk is underestimated. This negative interaction of risk which is caused by the non-additivity of the value function  $V$ . The following proposition classifies the functions  $V$  for which the approximation error is zero everywhere.



**Figure 1: Unsatisfactory approximation of true value changes by the sum of market and credit value changes.** For this figure we use  $V(a, e) = -a \cdot e$  and take the reference scenario  $a_0 = e_0 = 0.5$  which is in the back left corner. Compared to the reference scenario, the value change in the scenario  $(5.5, 5.5)$  is  $\Delta V(5.5, 5.5) = -30$ , which is shown in the front right corner. A move of the credit risk factor  $a$  from its reference value 0.5 to 5.5 causes a value change  $\Delta C(5.5) = -2.5$ , which is realized in the scenario  $(5.5, 0.5)$  in the front left corner. A move of the market risk factor  $e$  from its reference value 0.5 to 5.5 also causes a value change  $\Delta M(5.5) = -2.5$ , which is realized in the scenario  $(0.5, 5.5)$  in the back right corner. The approximation  $\Delta C(5.5) + \Delta M(5.5)$  is -5, which is represented by the point above the surface in the front right corner. The approximation overestimates the true value change of -30 by an amount of 25. The amount of overestimation is represented by the vertical line connecting the true integrated risk  $\Delta V(5.5, 5.5)$  to the approximation  $\Delta C(5.5) + \Delta M(5.5)$ .

**Proposition 1.** *The approximation is exact, that is  $\Delta V(a, e) = \Delta C(a) + \Delta M(e)$ , if and only if  $V$  has the form*

$$V(a, e) = V_1(a) + V_2(e). \quad (3)$$

*In this case the portfolio is separable into two subportfolios, one depending only on credit risk factors, the other depending only on market risk factors.*

This proposition is technically easy but conceptually important. In particular the “only if” part is interesting. Linear value functions  $V$  fulfil condition (3) and are therefore exactly approximated. More generally, smooth possibly non-linear functions with  $\frac{\partial^2 V}{\partial a \partial e} = 0$  everywhere are exactly approximated. In the Appendix we provide a version of Proposition 1 for the smooth case. That proposition shows that for smooth  $V$  with  $\frac{\partial^2 V}{\partial a \partial e} \neq 0$ ,  $D$  takes both positive and negative values.

Now going from valuation to risk assessment, the properties of the value change functions in various scenarios  $(a, e)$  carry over to risk measures and risk capital. Define the functions  $A$  and  $E$  on  $\Omega_A \times \Omega_E$  by  $A(a, e) := a$  and  $E(a, e) := e$ . If the parameter space  $\Omega_A \times \Omega_E$  is equipped with a probability measure, the functions  $A$  and  $E$  can be regarded as random vectors, and the functions  $V$ ,  $\Delta V$ ,  $\Delta C$ ,  $\Delta M$  etc. can be regarded as random variables. The random vector  $A$  describes the credit risk factors and has the marginal distribution by integrating over the market risk factors. The random vector  $E$  describes the market risk factors and has the marginal distribution by integrating over the market risk factors. Credit, market, and integrated risk can be written as

$$\begin{aligned}\Delta C &= V(A, e_0) - V(a_0, e_0) \\ \Delta M &= V(a_0, E) - V(a_0, e_0) \\ \Delta V &= V(A, E) - V(a_0, e_0)\end{aligned}$$

These functions can be identified with random variables describing the profit changes resulting in a pure credit risk, pure market risk, and an integrated analysis. They are random variables on the same probability space and can therefore be compared by any one risk measure  $\rho$ .

Defining  $\Delta C$  and  $\Delta M$  on the basis of the marginal distributions conforms to the intuition that in a pure credit (resp. market) risk analysis the distribution of credit (resp. market) risk factors is determined empirically and without reference to the market (resp. credit) risk factors, and that in the subsequent evaluation of the portfolio the market (credit) risk factors are assumed to have constant values  $e_0$  (resp.  $a_0$  chosen to exclude defaults).<sup>1</sup> To these random variables one can apply any coherent risk measure  $\rho$ .<sup>2</sup> The  $\rho(\Delta C)$  we get is the  $RC_c$  of eq. (1). Similarly  $\rho(\Delta M) = RC_m$ .

We measure the effect of an integrated analysis of market and credit risk by

$$I := \frac{\rho(\Delta V)}{\rho(\Delta C) + \rho(\Delta M)},$$

which we define if  $\rho(\Delta C) + \rho(\Delta M) > 0$  and  $\rho(\Delta V) \geq 0$ . In case of negative inter-risk interaction  $I > 1$ : integrated risk is larger than the sum of market

<sup>1</sup>If  $\Delta C$  (resp.  $\Delta M$ ) were defined on basis of the *conditional* distribution of credit (resp. market) risk factors given the values  $e_0$  (resp.  $a_0$ ), i.e.,  $\Delta C = (V|E = e_0) - V(a_0, e_0)$  (resp.  $\Delta M = (V|A = a_0) - V(a_0, e_0)$ ), then even for a linear portfolio and normally distributed risk factors integrated risk might be larger than the sum of credit and market risk. Proposition 2 does not hold for  $\Delta C$  and  $\Delta M$  defined on basis of the conditional distributions. This was pointed out to us by Piergiorgio Alessandri.

<sup>2</sup>Applying risk measures to value change functions, rather than to value functions, implies translation invariance: We have  $\rho(\Delta(V + \lambda)) = \rho(\Delta V)$  for arbitrary real numbers  $\lambda$ , rather than translation covariance,  $\rho(V + \lambda) = \rho(V) - \lambda$ . Readers preferring the application of risk measures only to value functions can read  $\rho(V) + V(a_0, e_0)$  instead of  $\rho(\Delta V)$ . The risk integration index has then to be defined by  $I = (\rho(V) + V(a_0, e_0)) / (\rho(C) + \rho(M) + 2V(a_0, e_0))$ .

risk and credit risk.  $I = 1.2$  means that total risk is 20% larger than the sum of credit and market risk. If  $I < 1$  we have positive interaction of credit and market risk. Integrated risk will be smaller than the sum of credit and market risk.

**Proposition 2.** *In the case of benign interaction of risk ( $D \geq 0$ ) separate analysis of market and credit risk overestimates true risk:*

$$\rho(\Delta V) \leq \rho(\Delta C) + \rho(\Delta M). \quad (4)$$

*This holds for all sub-additive risk measures  $\rho$ . Otherwise, in the case of malign interaction of risk ( $D < 0$  somewhere), there exists a coherent risk measure  $\rho$  for which separate analysis of market and credit risk underestimates true risk:*

$$\rho(\Delta V) > \rho(\Delta C) + \rho(\Delta M). \quad (5)$$

Propositions 1 and 2 together establish the inverse of the argument in the introduction. The Conclusion (“Under all subadditive risk measures the risk total risk is smaller or at most equal to the sum of market risk and credit risk.”) implies Premise 2 (“The portfolio is separable into a credit subportfolio and a market subportfolio.”)

Portfolios where credit and market risk are separated into different subportfolios were considered by Dimakos and Aas [2004] and Rosenberg and Schuermann [2006]. In this case  $V$  is of the form  $V(a, e) = V_1(a) + V_2(e)$ . For such a portfolio by Proposition 1 the approximation is exact, i.e.,  $\Delta V(a, e) = \Delta C(a) + \Delta M(e)$  and  $\Delta V = \Delta M + \Delta C$ . Thus  $\rho(\Delta V) = \rho(\Delta C + \Delta M) \leq \rho(\Delta C) + \rho(\Delta M)$  and  $I > 0$  for any subadditive risk measure  $\rho$ . This implies that inter-risk interaction is always positive for a portfolio with credit and market risk separated into different subportfolios. Thus the determination of risk capital that relies on the sum of risk capital for market risk and risk capital for credit risk will necessarily be conservative. Because they only consider portfolios separable into market and credit subportfolios the authors observe diversification effects from an integrated analysis of market and credit risk. If there is interaction between credit and market risk such a separation of risk-types into subportfolios is not possible. This is the situation we consider.

### 3 A toy example of underestimation of the true risk

As an example where the need for an integrated analysis of market and credit risk is obvious and where true risk is underestimated under the current regulatory paradigm we now analyze foreign currency loans. In order to understand the risk underestimation effect for this particular example we

first use a toy model that is stripped to the bare essentials to reveal the fundamental mechanisms.

Foreign currency loans have come to the attention (and to the concern) of supervisory authorities because these instruments have recently become highly popular among private households to take out home mortgages. This form of mortgage financing has been especially popular in Austria and in Central and Eastern Europe. Foreign currency loans can be seen as a carry-trade. In the carry-trade, an investor borrows money from one country, where the borrowing cost is low, and invests it in another country, where investments yield a high rate of return. The flip-side of the advantage of a low borrowing rate is an exchange rate risk. Since the debt service capacity of a borrower is a function of the exchange rate, his credit risk is a direct function of market risk factor changes. Foreign currency loans are therefore a clear case where market and credit risk factors have to be studied simultaneously.

To formalize a foreign currency loan in a toy model, consider a single obligor who has taken out a Swiss Franc loan of 1 Euro. At the current exchange rate of  $f(0)$  this amounts to a swiss franc loan of  $1/f(0)$ , where  $f(0)$  is the home currency value of the foreign currency at time 0. After one year the loan expires and the payment obligation is  $f(1)/f(0) =: e$ . We assume that the market risk factor  $e$  can vary in the interval  $(0, \infty)$  and, for the sake of the toy example, that the interest rate is zero. Without further specifications assume that the obligor's EUR payment ability at the expiry of the loan is  $a$ , and that this credit risk factor  $a$  can vary in the interval  $[0, \infty)$ .

The value of the position to the bank is zero if the payment ability  $a$  is greater or equal to the payment obligation  $e$ . If  $e$  is larger than  $a$ , the value of the position is  $a - e$ , which is negative. So the portfolio value function is

$$V(a, e) := \min(a, e) - e = -\max(e - a, 0). \quad (6)$$

Now let us fix some reference scenario  $(a_0, e_0)$ . Credit risk, the profit or loss of the bank arising from moves in the credit risk factor  $a$  alone, assuming the payment obligation of the obligor will have the value  $e_0$  with certainty, is

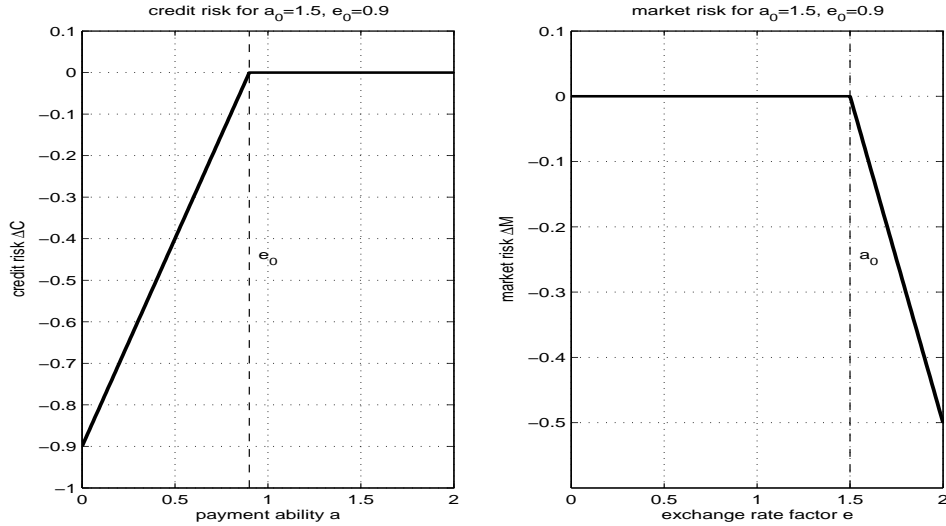
$$\Delta C(a) := V(a, e_0) - V(a_0, e_0) = -\max(e_0 - a, 0) + \max(e_0 - a_0, 0).$$

The profit or loss of the bank arising from moves in the market risk factor  $e$  alone, assuming the payment ability of the obligor will have the value  $a_0$  with certainty, is

$$\Delta M(e) := V(a_0, e) - V(a_0, e_0) = -\max(e - a_0, 0) + \max(e_0 - a_0, 0).$$

Assuming no defaults are possible would amount to choosing  $a_0 = \infty$ . But any other choice of  $a_0$  would also be possible. The smaller  $a_0$  the more

defaults will occur in the market risk scenarios. This increase market risk and decreases the negative inter-risk diversification effect. Still it is justified to call this a market risk analysis, because the credit risk factor is assumed to be constant and therefore is not a source of uncertainty.



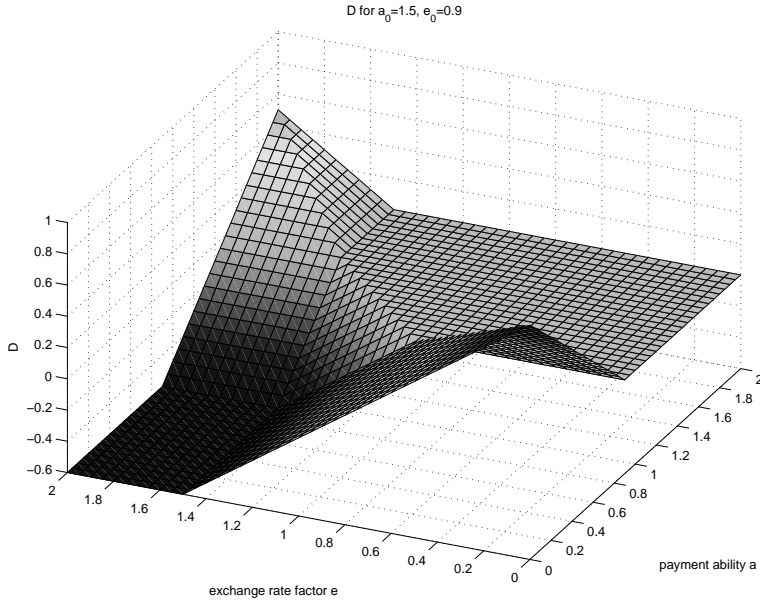
**Figure 2:** Credit risk  $\Delta C(a)$  (left) and market risk  $\Delta M(e)$  (right) for  $a_0 = 1.5, e_0 = 0.9$ .

Figure 2 plots credit risk  $\Delta C$  as a function of  $a$  (left) and market risk  $\Delta M$  as a function of the market risk factor  $e$  (right). Credit risk has the payoff profile of a short put on the payment ability  $a$  with strike  $e_0$ , which reflects Merton's key idea of structural credit risk models, regarding a loan as short put on the payment ability. Market risk has the payoff profile of a short call on the exchange rate  $e$  with strike  $a_0$ .

Does the separate calculation of credit and market risk overestimate or underestimate integrated risk? Figure 3 shows plots of the function  $D$  for  $a_0 = 1.5$  and  $e_0 = 0.9$ . The function  $D$  is negative in some regions. For scenarios in this region, integrated risk is larger than the sum of credit plus market risk. This is an example for negative interaction of credit and market risk. One can easily show analytically that  $D$  is negative in some region whenever  $a_0 \neq e_0$ . Only in the special case  $a_0 = e_0$  is  $D$  everywhere non-negative and an integrated analysis always leads to lower risk capital than a separate analysis.

## 4 A real world example

We have analyzed the logic of risk underestimation effects in theory and within the context of a toy example of a foreign currency loan. But do these



**Figure 3:** Plot of the function  $D$  in the toy model for  $a_0 = 1.5, e_0 = 0.9$ .

effects matter in real world examples? We want to use the last section to extend the toy model to a real world model that can be brought to the data. This analysis will give us some insight into the possible quantitative dimension of the problem.

Consider a portfolio of foreign currency loans with  $N$  obligors indexed by  $i = 1, \dots, N$ . All loans are underwritten at the initial time  $t = 0$ . In order to receive the home currency amount  $l_i$  an obligor takes a loan of  $l_i/f(0)$  units in the a foreign currency. The bank borrows  $l_i/f(0)$  units of the foreign currency on the interbank market. After one period, at time  $t = 1$ , which we take to be one year, the loan expires and the bank repays the foreign currency on the interbank market with an interest rate  $r_f$  and it claims from the customer a home currency amount which is exchanged at the rate  $f(1)$  to the foreign currency amount  $(l_i/f(0))(1 + r + s_f)$ , which is the original loan plus interest  $r_f$  rolled over from four quarters plus a spread  $s_f$ . So the customer's payment obligation to the bank at time 1 in home currency is

$$o_i = l_i(1 + r_f) f(1)/f(0) + l_i s_f f(1)/f(0). \quad (7)$$

The first term on the right hand side is what the bank has to repay on the interbank market, the second term is the spread profit of the bank. For a home currency loan the payment obligation would be  $o_i = l_i(1 + r_h + s_h)$ , where  $r_h$  is the interest rate in the home currency and  $s_h$  is the spread to be paid by the customer on a home currency loan. Whether an obligor will be able to meet this obligation depends on his payment ability

$a_i$ . Like in a structural credit risk model, we assume that an obligor defaults if his payment ability at the end of the period is smaller than his payment obligation.

**Assumption 1.** *Obligors default in case their payment ability  $a_i$  at the expiry of the loan is smaller than their payment obligation  $o_i$ . In case of default the customer pays  $a_i$  instead of  $o_i$ .*

The profit of the bank with obligor  $i$  is therefore

$$V_i := \min(a_i, o_i) - l_i(1 + r_f)f(1)/f(0). \quad (8)$$

$f(0)$  is the known exchange rate at time  $t = 0$ ,  $f(1)$  and  $r$  are random variables. In the profit function  $v_i$  the first term is what the obligor repays and the second term is what the bank has to pay on the interbank market.

We model the ability of an obligor to repay his obligations as a function of macroeconomic conditions and an idiosyncratic risk component. The form of our payment ability process resembles firm value process in the model of Merton [1974] but it is adapted to incorporate the macroeconomic influence as in Pesaran et al. [2005].

**Assumption 2.** *The payment ability at final time 1 for each single obligor  $i$  is distributed according to*

$$a_i(1) = a_i(0) \cdot \frac{GDP(1)}{GDP(0)} \cdot \epsilon, \quad (9)$$

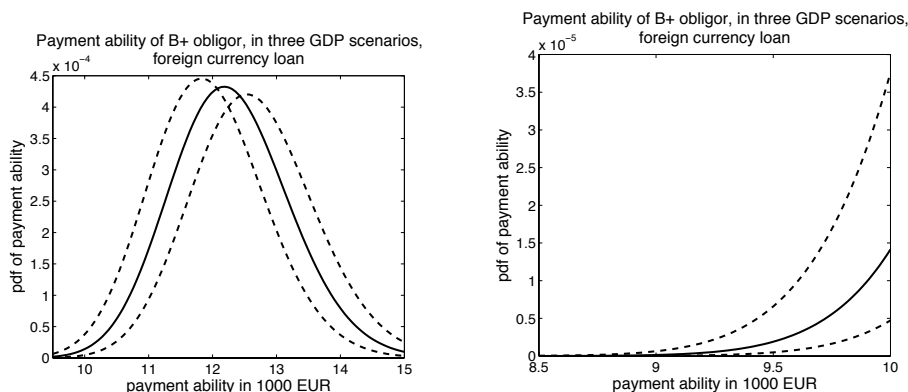
$$\log(\epsilon) \sim N(\mu, \sigma) \quad (10)$$

where  $a(0)$  is a constant, and  $\mu = -\sigma^2/2$  ensuring  $\mathbb{E}(\epsilon) = 1$ . For different obligors the realisations  $\epsilon_i$  are independent of each other and of GDP.

$GDP(0)$  is the known GDP at time  $t = 0$ ,  $GDP(1)$  is a random variable. The distribution of  $\epsilon_i$  reflects obligor specific random events, like losing or changing job. The support of  $\epsilon_i$  is  $(0, \infty)$  reflecting the fact that the amount  $a_i$  available for repayment of the loan cannot be less than zero if the obligor has no lines of credit open with the bank. Since the expected value of  $\epsilon_i$  is one and  $\epsilon$  is independent of GDP, the expectation of  $a_i(1)$  is  $a_i(0)$  times the expectation of  $GDP(1)/GDP(0)$ . Pesaran et al. [2005] use a model of this type for the returns of firm value. Assumption 2 amounts to taking in their model the predictable mean of the log-returns to be  $\log(GDP(1)/GDP(0))$ . A GDP increase shifts the payment ability distribution to the right. This is shown in Fig. 4. It increases distance to default and reduces default probabilities, provided the payment obligation is unchanged.

Assuming that for different customers the realizations of  $\epsilon_i$  are independent is the doubly stochastic hypothesis.<sup>3</sup> Conditional on the path of

<sup>3</sup>See Duffie and Singleton [2003]. Note also that there is some empirical evidence that the doubly stochastic hypothesis might be violated, cf. Das et al. [2007].



**Figure 4:** Plots of density function of the payment ability distribution, with GDP equal to its expected value (solid line), and GDP equal to  $\pm 3$  standard deviations. We observe (1) GDP increases lead to an increase in all quantiles, (2) GDP increases broaden the density function of the payment ability. The right hand plot is an enlargement of the left hand tail.

macro and market risk factors which determine the default intensities of all customers, customer defaults are independent.

The initial payment ability  $a_i(0)$  is a customer specific parameter determined in the loan approval procedure. For example, to be on the safe side the bank can extend loans only to customers with  $a_i(0)$  equal to 1.2 times the loan amount. This extra margin is taken into account in the rating. From a rating system the bank determines the default probability  $p_i$  of the customer on the expected payment obligation. In the loan approval procedure both the present payment ability  $a_i(0)$  and the rating (implying the default probability) are determined. They are input to our valuation model.

The payment ability distribution must satisfy the following condition:

$$p_i = P[a_i(1) < o_i]. \quad (11)$$

$a_i(1)$  is a function of  $\sigma$  and  $o_i$  is a function of the spreads. Spreads are set to achieve some target expected profit for each loan:

$$\mathbb{E}(V_i(\sigma, s)) = EP_{\text{target}}, \quad (12)$$

where  $v_i$  is the profit with obligor  $i$  and  $EP_{\text{target}}$  is some target expected profit. The two free parameters  $\sigma$  and  $s$  ( $s_f$  resp.  $s_h$ ) are determined from these two conditions.

How do credit and market risk factors interact in this model? At the end of the period, at time 1, after the obligor has paid the bank, and the bank has met its obligation at the interbank market, the bank has a net open foreign currency position  $s_f l_i f(1)/f(0)$  for obligor  $i$ . This is the only part of the position for which current regulation requires market risk capital. Default

risk on the other hand is determined by the probability that payment ability falls below payment obligation. This is a function of both the interest rate and the exchange rate. Thus default risk is a function of market risk factors. Therefore an integrated risk analysis is necessary.

To model the probability law of risk factors we use the GVAR time series model due to Pesaran et al. [2006]. The GVAR model is an error correction model that allows a parsimonious modelling of economic interdependence between countries or regions. This is exactly what we need in terms of risk factors, which involve exchange rate, interest rates and macroeconomic interactions between Austria and Switzerland. The basic idea of GVAR modelling is that it allows to build the global model from separately estimated country models with foreign variables entering the equation as weakly exogenous trade weighted averages. Country models can be estimated separately and stacked into a global model without reestimating the parameters. We estimate a GVAR model for Switzerland and Austria and include their three most important trading partners Germany, Italy and France as well as the most important trading partner of Germany, the US. The variables we consider for each country are real GDP, the three month LIBOR interest rate, and the exchange rate to the US dollar. Using the estimated parameters and the distributional assumptions of the model based on quarterly data from 1980q1 to 2005q4 we simulated one year ahead paths of the relevant risk factors and use our model assumptions (equations (7) to (9)) to simulate the profit distribution for the loan portfolio. For technical details we refer interested readers to Pesaran et al. [2006]<sup>4</sup>.

The distribution of the profit  $V$  (cf. eq. (8)) was calculated by a Monte Carlo simulation of 100 000 draws from the distribution of market and macro risk factors  $f(1)$ ,  $GDP(1)$ , and  $r$ . In each macro scenario defaults of the customers' payment abilities were determined by draws from the distribution of the payment ability process (9). The relative importance of GDP shocks versus idiosyncratic shocks is displayed in Fig. 4. The distribution of the macro risk factors was estimated from quarterly data 1989–2005 from the IFS of the International Monetary Fund. The estimated values for means and covariances of logarithms of the macro risk factors are given by the following Table.

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<sup>4</sup>To perform estimations and simulations we use our own R-implementation of the GVAR model based on Pesaran et al. [2000] and Pesaran et al. [2006]. Our implementation builds on work done by Zeugner [2006] who wrote a Matlab implementation of Pesaran et al. [2000].

	GDP	$r_{EUR}$	$r_{CHF}$	$f(1)$
mean	5.446	1.246	0.556	0.423
std. dev.	0.097	1.870	6.301	0.387
correlations	1.000	0.291	0.217	-0.040
		1.000	0.519	0.140
			1.000	0.007
				1.000

Let the portfolio be given with  $N = 100$  loans of  $l_i = \text{€}10\,000$  taken out in CHF by customers in the rating class B+, corresponding to a default probability of  $p_i = 2\%$ , or in rating class BBB+, corresponding to a default probability of  $p_i = 0.1\%$ . Assume that the bank extends loans only to customers with  $a_i(0)$  equal to 1.2 times the loan amount.

Our portfolio is of course still stylized because it assumes that all loans are underwritten at the initial time 0 and simultaneously expire at time 1. This simplification is however not essential to focus on the key question we have in mind here: Can we expect negative risk interaction to be quantitatively negligible or not?

The spreads  $s_f$  and  $s_h$  for each rating class were set in such a way that a target expected profit of  $e\,160$  on each loan is achieved, which amounts to a 20% return on an assumed capital charge of  $e\,800$  for a loan of  $\text{€}10\,000$ . The resulting spreads are:

rating	loan type	$\sigma$	spread [bp]
BBB+	home	0.0491	160.15
B+	home	0.0736	165.62
BBB+	foreign	0.0363	162.29
B+	foreign	0.0755	168.97

Note that in the same rating class spreads for FX loans are slightly higher than for home currency loans. This reflects the higher risk of FX loans, emerging from the subsequent analysis (compare Tables 1 and 2). Additionally, in order to achieve e.g. BBB+ for a home currency loan customer a  $\sigma$  of 0.0491 is sufficient, whereas a FX loan customer needs to achieve a much smaller  $\sigma$  of 0.0363. In other words, a customer with a given standard deviation  $\sigma$  in his payment ability will be in a higher rating class for a home currency loan, and in a lower rating class for an FX loan. Note that calculation of spreads only uses the expectation value of the profit distribution. If the calculation of spreads also included the cost of risk capital to cover

unexpected losses, spreads of FX loans would be even higher in order to cover the large unexpected losses possible for the FX loans (see Tables 1 and 2).

Does the separate calculation of credit and market risk capital overestimate or underestimate integrated risk capital? The market risk factors are  $e := (GDP(1), r_f, f(1))$  for the foreign currency loans and  $e := (GDP(1), r_h)$  for the home currency loans, the credit risk factors are  $a := (\epsilon_i)_{i=1, \dots, N}$ , and the portfolio value function is  $V = \sum_{i=1}^N V_i$ . As reference scenario we take the expected values  $e_0 := \mathbb{E}(e)$  of the market risk factors and  $a_0 := (\infty)_{i=1, \dots, N}$ , which implies that no obligor defaults.

We compare the distributions of integrated risk  $\Delta V(a, e)$  to the sum  $\Delta C(a) + \Delta M(e)$  of the distributions of market and credit risk by their Expected Shortfall (ES) at different quantiles  $\alpha$ .<sup>5</sup> In order to exclude non-subadditivity of the risk measure as a possible explanation for the the negative inter-risk diversification effect we calculate risk capital intended to cover unexpected losses as measured by Expected Shortfall (ES). For a profit loss distribution  $X$  risk capital is

$$RC_\alpha(X) := \mathbb{E}(X) - ES_\alpha(X), \quad (13)$$

where  $ES_\alpha$  is Expected Shortfall at some confidence level  $\alpha$ , as defined e.g. in [Acerbi and Tasche, 2002, Def. 2.6]. Standard deviations of approximation errors of ES are calculated using the method of Manistre and Hancock [2005].

Table 1 displays the risk capital for market, credit, and integrated risk. The key results of the simulation are in the last two columns of Table 1 which display the index  $I$ . These indices indicate negative risk interaction consistently, for all quantiles  $\alpha$ , and in both rating classes. Integrated risk capital is significantly higher than the sum of credit and market risk capital. Separate analysis underestimates true risk by factors between 1.13 and 8.22 for BBB+ and factors between 1.43 and 7.59 for the B+ portfolio.

Note that for the BBB+ foreign currency loan portfolio pure credit risk is very small. The intuitive explanation is that pure credit risk depends just on the idiosyncratic risk factors  $\epsilon_i$ . Other parameters being equal, a high variance  $\sigma$  of the  $\epsilon_i$  implies high pure credit risk. For the BBB+ foreign currency loan portfolios the  $\sigma$  values are lowest (Table 3). This a consequence of the calibration conditions (11) and (12).

These dramatic effects clearly reflect a malign interaction of market and credit risk which cannot be captured by providing separately for market and credit risk capital. Holding separate risk capital for market and for credit risk is by far not sufficient to cover the true integrated risk capital. This does not come as a surprise. The main risk of foreign currency loans, namely

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<sup>5</sup>The distribution  $\Delta C(a)$  is generated from the unconditional distribution of  $a$ , and the distribution  $\Delta M(e)$  is generated from the unconditional distribution of  $e$ .

the danger of increased defaults triggered by adverse exchange rate moves, is neither captured by market risk nor by credit risk models.

Table 2 shows that negative risk interaction also occurs for home currency loans. This is explainable by the dependence of default rates on the home interest rate. Home interest rate changes are reflected in payment obligation changes. Therefore an increase of this market risk factor triggers an increase in default rates. But the effect for home currency loans is much smaller than for foreign currency loans. Separate analysis underestimates true risk by factors between 1.25 and 2.75, depending on quantile and rating class. Negative risk interaction is weaker because the payment obligation of home currency loans depends much less sensitively on market factor changes. Note that for the home currency loan portfolio pure market risk is zero. The reason is that under the assumption that no customer defaults profits of the bank are certain and do not depend on exchange or interest rates.

**Table 1: Risk capital for market, credit, and integrated risks of the foreign currency loan portfolio.** Market risk  $\Delta M$  (MR) assumes no defaults are possible and considers only value changes due to market risk factor changes. Credit risk  $\Delta C$  (CR) reflects the value changes due to credit risk factor changes disregarding the possibility that market risk factors could vary stochastically. Market risk factors are fixed at their expected values. Integrated risk  $\Delta V$  calculates the value change assuming both credit and market risk factors simultaneously influence the value of a position. The final column calculates the inter-risk diversification index  $I$ . Standard deviations are shown in brackets. Initial income  $a_i(0) = \text{€}12\,000$ .

rating	$\alpha$	MR no CR RC( $\Delta M$ )	CR no MR RC( $\Delta C$ )	Integrated MR&CR RC( $\Delta V$ )	risk interaction $I$
BBB+	10%	1 059 (3)	0 (0)	1 193 (32)	1.13
BBB+	5%	1 234 (4)	0 (0)	1 522 (64)	1.23
BBB+	1%	1 576 (8)	0 (0)	3 056 (32)	1.94
BBB+	0.5%	1 698 (10)	1 (0)	4 641 (637)	2.73
BBB+	0.1%	1 951 (21)	3 (2)	16 076 (3137)	8.22
B+	10%	1 102 (4)	795 (4)	2 711 (49)	1.43
B+	5%	1 285 (5)	1 022 (6)	4 420 (94)	1.92
B+	1%	1 641 (8)	1 523 (14)	11 201 (388)	3.54
B+	0.5%	1 768 (11)	1 730 (19)	15 658 (713)	4.48
B+	0.1%	2 032 (22)	2 257 (45)	32 568 (2921)	7.59

**Table 2: Risk capital for market, credit, and integrated risks of the home currency loan portfolio.** Market risk  $\Delta M$  (MR) assumes no defaults are possible and considers only value changes due to market risk factor changes. For home currency loans market risk is zero if no defaults can occur. Credit risk  $\Delta C$  (CR) reflects the value changes due to credit risk factor changes disregarding the possibility that market risk factors could vary stochastically. Market risk factors are fixed at their expected values. Integrated risk  $\Delta V$  calculates the value change assuming both credit and market risk factors simultaneously influence the value of a position. The final column calculates the risk interaction index  $I$ . Standard deviations are shown in brackets. Initial income  $a_i(0) = \text{€}12\,000$ .

rating	$\alpha$	MR no CR RC( $\Delta M$ )	CR no MR RC( $\Delta C$ )	Integrated MR&CR RC( $\Delta V$ )	risk interaction $I$
BBB+	10%	0 (0)	48 (1)	132 (2)	2.75
BBB+	5%	0 (0)	102 (2)	238 (3)	2.33
BBB+	1%	0 (0)	310 (5)	462 (6)	1.49
BBB+	0.5%	0 (0)	396 (8)	556 (8)	1.40
BBB+	0.1%	0 (0)	589 (17)	774 (17)	1.31
B+	10%	0 (0)	961 (5)	1 257 (6)	1.31
B+	5%	0 (0)	1 222 (7)	1 582 (9)	1.29
B+	1%	0 (0)	1 805 (16)	2 299 (19)	1.27
B+	0.5%	0 (0)	2 052 (21)	2 594 (26)	1.26
B+	0.1%	0 (0)	2 601 (47)	3 260 (55)	1.25

**Table 3: Sensitivity of results on the choice of initial income  $a_0$ .** The final column calculates the risk interaction indices  $I$  and  $I$ . The second line shows that a customer with  $a_i(0) = \text{€}11\,000$  cannot achieve rating BBB+ on a foreign currency loan.

$a_i(0)$	rating	currency	spread	$\sigma$	MR		CR		Integrated		risk	
					no CR RC( $\Delta M$ )	no MR RC( $\Delta C$ )	MR RC( $\Delta v$ )	MR&CR RC( $\Delta v$ )	interaction	$I$		
11 000	BBB+	home	160.06	0.0197	0	58	269					4.64
11 000	BBB+	foreign	impossible									
11 000	B+	home	162.53	0.0316	0	824	1 775					2.15
11 000	B+	foreign	166.66	0.0150	1 618	0	31 569					19.51
12 000	BBB+	home	160.15	0.0491	0	310	462					1.49
12 000	BBB+	foreign	162.29	0.0363	1 576	0	3 056					1.94
12 000	B+	home	165.62	0.0736	0	1 805	2 299					1.27
12 000	B+	foreign	168.97	0.0755	1 641	1 523	11 201					3.54
13 000	BBB+	home	160.22	0.0745	0	515	662					1.29
13 000	BBB+	foreign	162.29	0.0711	1 576	236	2 015					1.11
13 000	B+	home	168.27	0.1109	0	2 705	3 167					1.17
13 000	B+	foreign	171.58	0.1163	1 666	2 663	7 921					1.83

How sensitively do these results depend on the choice of initial payment ability  $a_i(0)$ ? In our model the only two exogeneous input parameters are  $a_0$  and the rating class (resp. default probability). Table 3 shows the results for initial payment ability  $a_i(0) = \text{€} 11\,000$  and  $a_i(0) = \text{€} 13\,000$  at a confidence level of  $\alpha = 1\%$ . We observe that for lower  $a_i(0)$  the negative integration effect is considerably stronger than for higher  $a_i(0)$ , but it persists for all  $a_i(0)$ . The second line of Table 3 shows that a customer with  $a_i(0) = \text{€} 11\,000$  cannot achieve rating BBB+ on a foreign currency loan. Even if  $\sigma$ , i.e. idiosyncratic variation, were zero the variation in the payment obligation is too large for the required default probability of 0.01%.

## 5 Conclusions

In this paper we challenge the traditional regulatory approach of dividing risks according to the familiar categories of market and credit risk. We argue that this approach is conceptually problematic because many portfolios are not separable into a market subportfolio and a credit subportfolio. We argue that as a consequence risk assessment and the calculation of regulatory capital can be seriously flawed. Only if a portfolio is separable into market and credit subportfolio, we can be sure that calculating regulatory capital independently for market and credit risk and adding up, we will always calculate an upper bound for the necessary risk capital. Only for separable portfolios the current regulatory approach is conservative. If portfolio positions depend *simultaneously* on market and credit risk factors the nature of the risk assessment problem changes. If for such a portfolio market and credit risk are calculated separately, this is based on a wrong portfolio valuation and leads to a wrong assessment of true portfolio risk. Using the example of foreign currency loans we show that under the current regulatory concepts we could have a serious underestimation effect of the true risk of such a portfolio.

From the point of view of regulators, it might be difficult to require all institutions to introduce integrated market and credit risk analysis tools. One possible option could be to offer institutions a choice between integrated and separate market and credit risk analysis, but to require low values of  $a_0$  in the separate analysis. This implies that in pure market risk analysis the payment ability of all obligors is assumed to be equal to a small value  $a_0$ . In contrast, current market risk regulation assumes default risk to be zero, amounting to  $a_0 = \infty$ . (Distinguish  $a_0$  from the initial payment ability  $a_i(0)$ .) With a small value of  $a_0$ , more defaults occur in the market risk analysis, and market risk capital increases. Accordingly, the risk underestimation is weakened and turns into a positive effect for  $a_0$  small enough. Such a regulatory approach could ensure that we have only overestimation of the true risk. It is conservative in the sense of rather overestimating than

underestimating total risk. Additionally, this approach creates an incentive for institutions to develop integrated market and credit risk models, which yield lower but still safe regulatory capital requirements.

Let us finally discuss an alternative interpretation of our results.<sup>6</sup> In the example of the foreign currency loan portfolio the exchange rate can be considered both a market and a credit risk factor. It is a market risk factor because it has an effect on the portfolio value in case no defaults happen. By the same token it is a credit risk factor because it has an effect on the size of default losses. If a risk factor affects both market and credit risk one basic assumption of our analysis in Section 2 fails: Credit risk factors are not separate from market risk factors. Imposing such a separation amounts to a modelling error in either the market or the credit risk model.

A proper model of credit risk has to take into account all risk factors which have an effect on default losses. For the foreign currency loan portfolio this means that the credit risk model has to take into account moves of the exchange rate. This amounts to a proper credit risk model to be in fact an integrated model. Similarly, a proper market risk model has to take into account market value changes caused by changes in default probabilities, default correlations or losses given default, even if default does not actually happen. This amounts to a proper market risk model to be in fact an integrated model.

In this interpretational framework our results show that an approximate credit risk analysis assuming fixed values of the market risk factors can dramatically underestimate true credit risk if market and credit risk interact. Both interpretations of our analysis imply that a separate calculation of pure market risk and pure credit risk is not an admissible approximation to integrated risk if market and credit risk interact.

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## A Proof of Proposition 1

If  $V$  has the form  $V(a, e) = V_1(a) + V_2(e)$ , we have

$$\begin{aligned}
 \Delta C(a) + \Delta M(e) &= V(a_0, e) + V(a, e_0) - 2V(a_0, e_0) \\
 &= V_1(a_0) + V_2(e) + V_1(a) + V_2(e_0) - 2V_1(a_0) - 2V_2(e_0) \\
 &= V_1(a) + V_2(e) - V_1(a_0) - V_2(e_0) \\
 &= \Delta V(a, e).
 \end{aligned}$$

Conversely, if  $\Delta V(a, e) = \Delta C(a) + \Delta M(e)$ , then  $V(a, e) = V(a_0, e) + V(a, e_0) - V(a_0, e_0)$  which equals  $V_1(a) + V_2(e)$  for  $V_1(a) := V(a, e_0) - V(a_0, e_0)$  and  $V_2(e) := V(a_0, e)$ .  $\square$

## B Generalization of Proposition 1 to the smooth case.

We consider a function  $V$  depending on one market risk factor  $e$  and one credit risk factor  $a$ .

**Proposition 3.** *If  $V$  has continuous second order derivatives the approximation error  $D(a, e)$  with respect to the reference scenario  $(a_0, e_0)$  can be calculated as*

$$D(a, e) = \int_{a_0}^a \int_{e_0}^e \frac{\partial^2 V}{\partial a \partial e}(x, y) dy dx. \quad (14)$$

*Assuming additionally the second derivative of  $V$  is continuous this implies the following: If  $\frac{\partial^2 V}{\partial a \partial e}(a_0, e_0) \neq 0$ , then within a neighbourhood of  $(a_0, e_0)$ ,  $D$  is negative in two opposite quadrants separated by  $(a_0, e_0)$  and it is positive in the other two opposite quadrants.*

We first prove the result for  $V$  1-dimensional market and one credit risk factors  $a$  and  $e$ .

$$\begin{aligned}
 V(a, e) &= \int_{a_0}^a \frac{\partial V}{\partial a}(x, e) dx + V(a_0, e) \\
 &= \int_{a_0}^a \left[ \int_{e_0}^e \frac{\partial^2 V}{\partial e \partial a}(x, y) dy + \frac{\partial V}{\partial a}(x, e_0) \right] dx + V(a_0, e) \\
 &= \int_{a_0}^a \int_{e_0}^e \frac{\partial^2 V}{\partial e \partial a}(x, y) dy dx + \int_{a_0}^a \frac{\partial V}{\partial a}(x, e_0) dx + V(a_0, e) \\
 &= \int_{a_0}^a \int_{e_0}^e \frac{\partial^2 V}{\partial e \partial a}(x, y) dy dx + V(a, e_0) - V(a_0, e_0) + V(a_0, e).
 \end{aligned}$$

Thus,

$$\begin{aligned}
 D(a, e) &= V(a, e) - V(a, e_0) - V(a_0, e) + V(a_0, e_0) \\
 &= \int_{a_0}^a \int_{e_0}^e \frac{\partial^2 V}{\partial e \partial a}(x, y) dy dx.
 \end{aligned} \quad (15)$$

□

**Proposition 4.** *Let  $V : \Omega_A \times \Omega_E \rightarrow \mathbb{R}$  with reference scenario  $a_0 \in \Omega_A \subseteq \mathbb{R}^m$  and  $e_0 \in \Omega_E \subseteq \mathbb{R}^n$ . If  $V$  has continuous second order derivatives the approximation error  $d(a, e)$  with respect to the reference scenario  $(a_0, e_0)$  can be calculated as*

$$D(a, e) = \int_0^1 \int_0^1 \sum_{i=1}^m \sum_{j=1}^n \frac{\partial^2 V}{\partial e^j \partial a^i} (a_0 + (a - a_0)s, e_0 + (e - e_0)t) (a - a_0)^i (e - e_0)^j dt ds. \quad (16)$$

where superscripts refer to components of a vector.

Choose smooth paths  $\gamma_a : [0, 1] \rightarrow A$  and  $\gamma_e : [0, 1] \rightarrow E$  connecting  $a_0$  with some  $a \in A$  and  $e_0$  with some  $e \in E$ . Then we have

$$\begin{aligned} V(a, e) &= \int_0^1 \sum_{i=1}^m \frac{\partial V}{\partial a^i} (\gamma_a(s), e) \dot{\gamma}_a^i(s) ds + V(a_0, e) \\ &= \int_0^1 \sum_{i=1}^m \int_0^1 \sum_{j=1}^n \frac{\partial^2 V}{\partial e^j \partial a^i} (\gamma_a(s), \gamma_e(t)) \dot{\gamma}_a^i(s) \dot{\gamma}_e^j(t) dt ds + \dots \\ &\quad + \int_0^1 \sum_{i=1}^m \frac{\partial V}{\partial a^i} (\gamma_a(s), e_0) \dot{\gamma}_a^i(s) ds + V(a_0, e) \\ &= \int_0^1 \int_0^1 \sum_{i=1}^m \sum_{j=1}^n \frac{\partial^2 V}{\partial e^j \partial a^i} (\gamma_a(s), \gamma_e(t)) \dot{\gamma}_a^i(s) \dot{\gamma}_e^j(t) dt ds + \dots \\ &\quad + V(a, e_0) - V(a_0, e_0) + V(a_0, e). \end{aligned}$$

Thus, we have Thus,

$$\begin{aligned} D(a, e) &= V(a, e) - V(a, e_0) - V(a_0, e) + V(a_0, e_0) \\ &= \int_0^1 \int_0^1 \sum_{i=1}^m \sum_{j=1}^n \frac{\partial^2 V}{\partial e^j \partial a^i} (\gamma_a(s), \gamma_e(t)) \dot{\gamma}_a^i(s) \dot{\gamma}_e^j(t) dt ds. \quad (17) \end{aligned}$$

This can be simplified to eq. (16) by choosing the paths  $\gamma_a(s) := a_0 + (a - a_0)s$  and  $\gamma_e(s) := e_0 + (e - e_0)t$ . □

## C Proof of Proposition 2

Since  $D = \Delta V - \Delta C - \Delta M \geq 0$  for all scenarios,  $\Delta V \geq \Delta M + \Delta C$  holds for the corresponding random variables. Monotonicity and subadditivity of a coherent risk measure  $\rho$  imply  $\rho(\Delta V) \leq \rho(\Delta C + \Delta M) \leq \rho(\Delta C) + \rho(\Delta M)$ . Conversely, assume there is some scenario  $(a^*, e^*)$  for which  $0 > D(a^*, e^*) = \Delta V(a^*, e^*) - \Delta C(a^*) - \Delta M(e^*)$ . Now consider the following risk measure  $\rho$ .

To each random variable  $F$  arising from a portfolio function  $f : \Omega_A \times \Omega_E \rightarrow \mathbb{R}$  by  $F := f(A, E)$  the risk number  $\rho(F) := -f(a^*, e^*)$  is assigned. This is a coherent risk measure. We have  $\rho(\Delta C) = \rho(V(A, e_0) - V(a_0, e_0)) = -V(a^*, e_0) + V(a_0, e_0)$ , and similarly for  $\Delta M$  and  $\Delta V$ . Thus  $\rho(\Delta V) = -V(a^*, e^*) + V(a_0, e_0) > -V(a^*, e_0) + V(a_0, e_0) - V(a_0, e^*) + V(a_0, e_0) = \rho(\Delta C) + \rho(\Delta M)$ .  $\square$